

Kolmogorov complexity and the geometry of Brownian motion

Willem L. Fouché

Department of Decision Sciences,

School of Economic Sciences

University of South Africa, PO Box 392, 0003 Pretoria, South Africa

fouchwl@gmail.com

Abstract

In this paper, we continue the study of the geometry of Brownian motions which are encoded by Kolmogorov-Chaitin random reals (complex oscillations). We unfold Kolmogorov-Chaitin complexity in the context of Brownian motion and specifically to phenomena emerging from the random geometric patterns generated by a Brownian motion.

Key words: Kolmogorov complexity, Martin-Löf randomness, Brownian motion, countable dense random sets, descriptive set theory.

1 Introduction

Finally, we would like to comment on the hidden role of Kolmogorov complexity in the real life of classical computing

The inherent tension, incompatibility of shortest descriptions with most-economical algorithmical processing, is the central issue of any computability theory.

The place-value notation of numbers that played such a great role in the development of human civilizations is the ultimate system of short descriptions that bridges the abyss. Kolmogorov complexity goes far beyond this point. ([Manin 2010] p 327.)

It is well-known that the notion of randomness, suitably refined, goes a long way in dealing with this tension. (See, for example, [Chaitin 1987, Martin-Löf 1966, Nies 2008].) In this paper, we continue to explore this interplay between short descriptions and randomness in the context of Brownian motion and its associated geometry. In this way one sees how random phenomena associated with the geometry of Brownian motion, are implicitly enfolded in each real number which is complex in the sense of Kolmogorov. These random phenomena range from fractal geometry, Fourier analysis and non-classical noises in quantum physics.

We study in this paper algorithmically random Brownian motion, the representations of which were also called *complex oscillations* in [Fouché(1) 2000, Fouché(2) 2000] for example. This terminology was suggested to the author by the following Kolmogorov theoretic interpretation of this notion by Asarin and Prokovskii [Asarin and Prokovskii 1986], who are the pioneers of this theme. One can characterise a Brownian motion which is algorithmically random (or, equivalently Martin-Löf random) as an effective and uniform limit of a sequence (x_n) of “finite random walks”, where, moreover, each x_n can be encoded by a finite binary string s_n of length n , such that the (prefix-free) Kolmogorov complexity, $K(s_n)$, of s_n satisfies, for some constant $d > 0$, the inequality $K(s_n) > n - d$ for all values of n .

Two other characterisations of the class of complex oscillations were developed by the author in [Fouché(1) 2000] and [Fouché(2) 2000]. In [Fouché(1) 2000] the class of complex oscillations were described in terms of effective subalgebras of the Borel σ -algebra on $C[0, 1]$ (with the supremum norm topology). The central idea here was to effectivise the Donsker invariance principle, i.e., to focus on Brownian motion as a scaling limit of finitary random walks. In [Fouché(2) 2000] they are shown to be

exactly those real-valued continuous functions on the unit interval that can be computed from Martin-Löf-random reals (relative to the Lebesgue measure) by means of an associated Franklin-Wiener series. Here, the guiding motif was the fact that one can, as is well-known, also think of Brownian motion as a linear superposition of deterministic oscillations with normally distributed random amplitudes. Further applications and developments of this idea can be found in the papers [Fouché 2008, Fouché 2009, Hoyrup and Rojas 2009, Potgieter 2012, Kjos-Hanssen and Nerode 2009]. A sharper version, from a computational point of view, of the main result in [Fouché(2) 2000] has recently been developed by George Davie and the author [Davie and Fouché 2012].

Countable dense random sets arise naturally in the theory of Brownian motion [Tsirelson 2006], in non-classical noises [Tsirelson 2004] and the understanding of percolation phenomena in statistical physics (see [Tsirelson 2004], [Camia, Fontes and Newman 2005], for example). It is an interesting fact that the study of countable dense random sets quite naturally brings one in contact with studying random processes over spaces which are not even Polish. One has to do probability theory over orbit spaces under the action of the group S_∞ , which is the symmetry group of a countable set, on the space of all injections of \mathbb{N} into the unit interval. These are examples of what Kechris [Kechris 1999] referred to as singular spaces of Borel cardinality F_2 .

In [Tsirelson 2006] Tsirelson develops a very powerful approach to random processes over these singular spaces and his results imply that the Kechris-singularity manifests in very concrete and interesting statistical properties of countable dense random sets and new aspects of Brownian motion.

Tsirelson [Tsirelson 2006] shows that the minimizers of a Brownian motion are, in the language of [Tsirelson 2004], instances of so-called stationary local random dense countable sets over the white noise and that they play a pivotal role in the understanding of non-classical noises.

This work suggested to the author the problem of constructing the minimizers of Brownian motion directly from an unbiased coin-tossing experiment. This can be seen as an extension of [Fouché(2) 2000] where a generic Brownian motion was constructed from a generic point in the unit interval. We shall again adopt the viewpoint of Kolmogorov complexity to define what we mean by the word generic. In this way, we shall be able to find Σ_3^0 definitions, within the arithmetical hierarchy, for countable dense random sets, which can be considered to be “generic” countable dense sets of reals and moreover symmetrically random over white noise. We provide an explicit computable enumeration of the elements of such sets relative to Kolmogorov-Chaitin-Martin-Löf random real numbers. This opens the way to relate certain non-classical noises to Kolmogorov complexity. For example, the work of the present paper enables one to represent Warren’s splitting noise (see [Tsirelson 2004]) directly in terms of infinite binary strings which are Kolmogorov-Chaitin-Martin-Löf random. This line of thought will also be pursued in a sequel to this paper.

In this sequel to this paper, we shall study the images of certain Π_2^0 perfect sets of Hausdorff dimension zero under a complex oscillation. We have given a sketch in the extended abstract [Fouché 2009] of a proof that there are instances of such sets where such images under complex oscillations have elements all of which are linearly independent over the field of rational numbers. In Fourier analysis, these sets are called sets of independence. (See, for example, Chapter 5 of Rudin’s book [Rudin 1960], pp 97-130.) We shall provide a generalisation of this result and show in fact that one can obtain sets via complex oscillations which are linearly independent over the field of recursive real numbers. Moreover, all the elements in these images are non-computable.

The author is grateful to the Department of Mathematics at the Corvinus University, Budapest, for hosting my frequent visits to the department and for sharing with me so much of the subtleties of measure theory and stochastic processes.

The research in this paper has been supported by the National Research Foundation (NRF) of South Africa and by the European Union grant agreement PIRSES-GA-2011-2011-294962 in Computable Analysis (COMPUTAL).

Many thanks are due to the referee whose remarks led to a significant strengthening of Theorem 5.

2 Preliminaries and statements of the main theorems

A Brownian motion on the unit interval is a real-valued function $(\omega, t) \mapsto X_\omega(t)$ on $\Omega \times [0, 1]$, where Ω is the underlying space of some probability space, such that $X_\omega(0) = 0$ a.s. and for $t_1 < \dots < t_n$ in the unit interval, the random variables $X_\omega(t_1), X_\omega(t_2) - X_\omega(t_1), \dots, X_\omega(t_n) - X_\omega(t_{n-1})$ are statistically independent and normally distributed with means all 0 and variances $t_1, t_2 - t_1, \dots, t_n - t_{n-1}$, respectively. We say in this case that the Brownian motion is parametrised by Ω . Alternatively, the map X defines a Brownian motion iff for $t_1 < \dots < t_n$ in the unit interval, the random vector $(X_\omega(t_1), \dots, X_\omega(t_n))$ is Gaussian with correlation matrix $(\min(t_i, t_j) : 1 \leq i, j \leq n)$.

It is a fundamental fact that any Brownian motion has a “continuous version”. This means the following: Write Σ for the σ -algebra of Borel sets of $C[0, 1]$ where the latter is topologised by the uniform norm topology. There is a probability measure W on Σ such that for $0 \leq t_1 < \dots < t_n \leq 1$ and for a Borel subset B of \mathbf{R}^n , we have

$$P(\{\omega \in \Omega : (X_\omega(t_1), \dots, X_\omega(t_n)) \in B\}) = W(A),$$

where

$$A = \{x \in C[0, 1] : (x(t_1), \dots, x(t_n)) \in B\}.$$

The measure W is known as the *Wiener measure*. We shall usually write $X(t)$ instead of $X_\omega(t)$.

In the sequel, we shall denote by $(0, 1)^\infty$ the Borel space consisting of the product of countably many copies of the unit interval and with Borel structure being given by the natural product structure which is induced by the standard Borel structure on the unit interval. We write $(0, 1)^\infty_\neq$ for the Borel subspace consisting of the infinite sequences in the unit interval which are pairwise distinct.

We write S_∞ for the symmetric group of a countable set (which we can take to be \mathbb{N}). We place on S_∞ the pointwise topology. We thus give S_∞ the subspace topology under the embedding of S_∞ into the Baire space $\mathbb{N}^\mathbb{N}$. The group S_∞ acts naturally (and continuously) on $(0, 1)^\infty_\neq$ as follows:

$$\sigma.(u_j : j \geq 1) := (u_{\sigma^{-1}(j)} : j \geq 1),$$

for all $(u_j) \in (0, 1)^\infty_\neq$ and $\sigma \in S_\infty$ (the logical action). The orbit space under this action is denoted by $(0, 1)^\infty_\neq / S_\infty$. We place a Borel structure on this space via the topology induced by the canonical mapping

$$\pi : (0, 1)^\infty_\neq \longrightarrow (0, 1)^\infty_\neq / S_\infty.$$

Let Ω be standard Borel space. A *strongly* random countable set in the unit interval is a measurable mapping $X : \Omega \rightarrow (0, 1)^\infty_\neq / S_\infty$ that factors through some (traditional) random sequence Y as shown:

$$\begin{array}{ccc} \Omega & \xrightarrow{Y} & (0, 1)^\infty_\neq \\ & \searrow X & \swarrow \pi \\ & (0, 1)^\infty_\neq / S_\infty & \end{array}$$

One can think of X as a random countable *set* induced via S_∞ -equivalence, by a random *sequence* Y , both in the unit interval. In the sequel, we shall sometimes denote the Borel space $(0, 1)^\infty_\neq / S_\infty$ by $CS(0, 1)$.

As noted by Tsirelson [Tsirelson 2006] the natural question as to whether any measurable $X : \Omega \rightarrow (0, 1)^\infty_\neq / S_\infty$ factors through some Y as above, is an open problem.

The following fundamental theorem of Tsirelson’s explains exactly what it means for two strongly random countable sets to be “statistically similar”.

Theorem 1 ([Tsirelson 2006]). *For standard measure spaces (Ω_1, P_1) and (Ω_2, P_2) , let, for $i = 1, 2$, there be, some P_i -measurable strongly random set $X_i : \Omega_i \rightarrow CS(0, 1)$ such that the induced probability distributions on $CS(0, 1)$ are the same, i.e., for every Borel subset Σ of $CS(0, 1)$ it is the case that*

$$P_1(X_1^{-1}(\Sigma)) = P_2(X_2^{-1}(\Sigma)).$$

Then there is a probability distribution \mathbb{P} on $\Omega_1 \times \Omega_2$ such that the marginal of \mathbb{P} to Ω_i is P_i , and moreover, for \mathbb{P} almost all $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$ it is the case that

$$X_1(\omega_1) = X_2(\omega_2).$$

The statement “the marginal of \mathbb{P} to Ω_i is P_i ”, means that for measurable $\Sigma_i \subset \Omega_i$ $i = 1, 2$:

$$\mathbb{P}(\Sigma_1 \times \Omega_2) = P_1(\Sigma_1);$$

and

$$\mathbb{P}(\Omega_1 \times \Sigma_2) = P_2(\Sigma_2).$$

We say in this case that the strongly random sets X_1 and X_2 are *statistically similar* relative to the probabilities P_1, P_2 and we simply write $X_1 \sim X_2$.

A strongly random countable set $X : \Omega \rightarrow CS(0, 1)$ is said to be *generic* relative to a probability measure P on Ω if the following is true:

If B is a Borel subset of the unit interval such that $\lambda(B) > 0$, then P -almost surely, $B \cap X \neq \emptyset$.

On the other hand, if $\lambda(B) = 0$, then P -almost surely, $B \cap X = \emptyset$. Equivalently, if C is a Borel set such that $\lambda(C) = 1$, then, almost surely, $X \subset C$.

Here we have written λ for the Lebesgue measure on the unit interval. Note that if X is generic and if $Y \sim X$, then Y too is generic.

A partial converse of this statement can be found in [Tsirelson 2006]: If X_1, X_2 are both strongly random, each satisfying what Tsirelson calls the “independence condition” relative to a probability measure P_i , and each being almost surely dense in the unit interval, then they are statistically similar provided they are both generic!! (Tsirelson 2006).

Write λ^∞ for the product measure on $(0, 1)^\infty$ which is the countable product of the Lebesgue measure λ on the unit interval and write Λ for the measure on $CS(0, 1)$ which is the pushout of λ^∞ under π . In other words, for a Borel subset Σ of $CS(0, 1)$,

$$\Lambda(\Sigma) = \lambda^\infty(\pi^{-1}\Sigma).$$

Write $U : (0, 1)^\infty \rightarrow CS(0, 1)$ for the strongly random set as defined by the following commutative diagram:

$$\begin{array}{ccc} (0, 1)^\infty & \xrightarrow{Id} & (0, 1)^\infty \\ & \searrow U & \swarrow \pi \\ & CS(0, 1) = (0, 1)^\infty / S_\infty & \end{array}$$

Then U is almost surely dense and generic. In statistics U is a model of an unordered uniform infinite sample. Moreover, it follows from the Hewitt-Savage theorem, that for every Borel subset Σ of $CS(0, 1)$, it is the case that

$$\Lambda(\Sigma) \in \{0, 1\}. \quad (1)$$

Note that Λ is non-atomic. Consequently, $CS(0, 1)$ is not a Polish space! We shall refer to the strongly random set U as the *uniform random set*.

If X is a continuous function on the unit interval, then a *local minimizer* of X is a point t such that there is some closed interval $I \subset [0, 1]$ containing t such that the function X assumes its minimum value on I at the point t . We denote by $\text{MIN}(X)$ the set of local minimizers of X . It is well-known that if X is a continuous version of Brownian motion on the unit interval, then $\text{MIN}(X)$ is almost surely a dense and countable set and that all the local minimizers of X are *strict*. This means that, for each closed subinterval I of the closed unit interval, there is a unique $\nu \in I$ where the minimum of X on I is assumed. This, as will be explained in this paper, has the implication that there is a subset Ω_0 of $C[0, 1]$ of full

Wiener measure such that one can define a measurable mapping $\min : C[0, 1] \supset \Omega_0 \rightarrow (0, 1)_{\neq}^{\infty}$ in such a way that the composition of \min with the projection π will define a measurable mapping $X \mapsto \text{MIN}(X)$. In the sequel this strongly random set will be denoted by MIN . To summarise, we have the following commutative diagram:

$$\begin{array}{ccc} C[0, 1] \supset \Omega_0 & \xrightarrow{\min} & (0, 1)_{\neq}^{\infty} \\ & \searrow \text{MIN} \quad \swarrow \pi & \\ & (0, 1)_{\neq}^{\infty} / S_{\infty} & \end{array}$$

The next theorem of Tsirelson (2006) says essentially that the local minimizers of a Brownian motion is a generic countable dense random set. (It is quite trivial to show that it satisfies the independence property.)

Theorem 2 [Tsirelson 2006]. *If X is a continuous version of Brownian motion on the unit interval and B is a Borel subset of the unit interval such that $\lambda(B) > 0$, then almost surely, $B \cap \text{MIN}(X) \neq \emptyset$. On the other hand, if $\lambda(B) = 0$, then almost surely, $B \cap \text{MIN}(X) = \emptyset$. In particular, if $\lambda(C) = 1$, then, almost surely, $\text{MIN}(X) \subset C$.*

It follows that any generic countable dense random set with the independence property will be statistically similar to the random set of minimizers of a Brownian motion. In particular

$$\text{MIN} \sim U. \quad (2)$$

The set of words over the alphabet $\{0, 1\}$ is denoted by $\{0, 1\}^*$. If $a \in \{0, 1\}^*$, we write $|a|$ for the length of a . If $\alpha = \alpha_0\alpha_1\ldots$ is an infinite word over the alphabet $\{0, 1\}$, we write $\overline{\alpha}(n)$ for the word $\prod_{j < n} \alpha_j$. We use the usual recursion-theoretic terminology Σ_r^0 and Π_r^0 for the arithmetical subsets of $\mathbb{N}^k \times \{0, 1\}^{\mathbb{N} \times l}$, $k, l \geq 0$. (See, for example, [Hinman 1978]). We again write λ for the Lebesgue probability measure on $\{0, 1\}$. For a binary word s of length n , say, we write $[s]$ for the “interval” $\{\alpha \in \{0, 1\}^{\mathbb{N}} : \overline{\alpha}(n) = s\}$. A sequence (a_n) of real numbers converges *effectively* to 0 as $n \rightarrow \infty$ if for some total recursive $f : \mathbb{N} \rightarrow \mathbb{N}$, it is the case that $|a_n| \leq (m+1)^{-1}$ whenever $n \geq f(m)$.

For any finite binary word a we denote its (prefix-free) Kolmogorov complexity by $K(a)$. Recall that an infinite binary string α is Kolmogorov-Chaitin complex if

$$\exists d \forall n \ K(\overline{\alpha}(n)) \geq n - d. \quad (3)$$

In the sequel, we shall denote this set by KC and refer to its elements as KC -strings. (See, e.g., [Chaitin 1987], [Martin-Löf 1966] or [Nies 2008] for more background.)

For $n \geq 1$, we write C_n for the class of continuous functions on the unit interval that vanish at 0 and are linear with slopes $\pm\sqrt{n}$ on the intervals $[(i-1)/n, i/n]$, $i = 1, \dots, n$. With every $x \in C_n$, one can associate a binary string $a = a_1 \cdots a_n$ by setting $a_i = 1$ or $a_i = 0$ according to whether x increases or decreases on the interval $[(i-1)/n, i/n]$. We call the sequence a the code of x and denote it by $c(x)$. The following notion was introduced by Asarin and Prokovskii in [Asarin and Prokovskii 1986].

Definition 1 *A sequence (x_n) in $C[0, 1]$ is complex if $x_n \in C_n$ for each n and there is a constant $d > 0$ such that $K(c(x_n)) \geq n - d$ for all n . A function $x \in C[0, 1]$ is a complex oscillation if there is a complex sequence (x_n) such that $\|x - x_n\|$ converges effectively to 0 as $n \rightarrow \infty$.*

The class of complex oscillations is denoted by \mathcal{C} . It was shown by Asarin and Prokovskii [Asarin and Prokovskii 1986] that the class \mathcal{C} has Wiener measure 1. In fact, they implicitly showed that the class corresponds exactly, in the broad context and modern language of Hoyrup and Rojas [Hoyrup and Rojas 2009], to the Martin-Löf random elements of the computable measure space ([Weihrauch 1999, Weihrauch 2000, Gács 2005])

$$\mathcal{R} = (C_0[0, 1], d, B, W), \quad (4)$$

where $C_0[0, 1]$ is the set of continuous functions on the unit interval that vanish at the origin, d is the metric induced by the uniform norm, B is the countable set of piecewise linear functions f vanishing at the origin with slopes and points of non-differentiability all rational numbers and where W is the Wiener measure.

For recent refinements of this result, the reader is referred to the work of Kjos-Hanssen and Szabados [Kjos-Hanssen and Szabados 2011]. They note that Brownian motion and scaled, interpolated simple random walks can be jointly embedded in a probability space in such a way that almost surely, the n -step walk is, with respect to the uniform norm, within a distance $O(n^{-\frac{1}{2}} \log n)$ of the Brownian path, for all but finitely many positive integers n . In the same paper, Kjos-Hanssen and Szabados show that, almost surely, their constructed sequence (x_n) of n -step walks is complex in the sense of Definition 1 and all Martin-Löf random paths have such an incompressible close approximant. This strengthens a result of Asarin [Asarin 1988], who obtained instead the bound $O(n^{-\frac{1}{6}} \log n)$.

The following theorem can be extracted from [Fouché(2) 2000]:

Theorem 3 *There is a bijection $\Phi : KC \rightarrow \mathcal{C}$ and a uniform algorithm that, relative to any KC -string α , with input a dyadic rational number t in the unit interval and a natural number n , will output the first n bits of the value of the complex oscillation $\Phi(\alpha)$ at t .*

The construction in [Fouché(2) 2000] of the complex oscillation $\Phi(\alpha)$ from a given $\alpha \in KC$ is as follows. Beginning with $\alpha \in KC$ we can construct a sequence of reals $\xi_0, \xi_1, \xi_{jn}, j \geq 1, 0 \leq n < 2^j$; the sequence is computable in α, j and n . Thereafter, we recursively find $x(n/2^j)$ for $n, j \in \mathbb{N}$ with $n \leq 2^j$ from the ξ -sequence by solving the equations

$$x(1) = \xi_0, \quad 2x\left(\frac{1}{2}\right) = \xi_0 + \xi_1,$$

and

$$2x\left(\frac{2n+1}{2^{j+1}}\right) = 2^{-j/2}\xi_{jn} + x\left(\frac{n+1}{2^j}\right) + x\left(\frac{n}{2^j}\right).$$

By the arguments in [Fouché(2) 2000], the complex oscillation $\Phi(\alpha)$ associated with a given $\alpha \in KC$ turns out to be the unique continuous function which assumes, for every dyadic rational d , the value $x(d)$. In this way, one can effectively compute any finite initial segment of the value of $\Phi(\alpha)$ at a given dyadic rational number from some initial segment of α . It also follows from the construction in [Fouché(2) 2000] that

$$\Phi(\alpha) = -\Phi(\hat{\alpha}). \quad (5)$$

Here $\hat{\alpha}$ denotes the binary string obtained from α by replacing each bit α_i of α by $1 - \alpha_i$.

The mapping Φ is also measure-preserving in the following sense: Let B be a Borel subset of $C[0, 1]$. Then

$$\lambda(\alpha \in KC : \Phi(\alpha) \in B) = W(B).$$

Let \mathcal{N} be the function that associates with every $x \in \mathcal{C}$, the set of local minimizers of x . We shall discuss the measurability and computability of \mathcal{N} in Section 4 of this paper. Thus \mathcal{N} is the restriction of MIN to \mathcal{C} . We then define the function

$$MLN : KC \longrightarrow (0, 1)_{\neq}^{\infty} / S_{\infty}$$

by

$$\alpha \mapsto \text{MIN}(\Phi(\alpha));$$

this means that the diagram

$$\begin{array}{ccc} KC & \xrightarrow{MLN} & (0, 1)_{\neq}^{\infty} / S_{\infty} \\ & \searrow \Phi & \nearrow \mathcal{N} \\ & \mathcal{C} & \end{array}$$

commutes. It follows from (1), (2) and the fact that Φ is measure-preserving, that, for every Borel subset Σ of $(0, 1)_{\neq}^\infty / S_\infty$:

$$\lambda(\alpha \in KC : \mathcal{MIN}(\alpha) \in \Sigma) \in \{0, 1\}.$$

(The zero-one law for the minimizers of complex oscillations.)

What is essentially at stake here is the Hewitt-Savage theorem together with the statistical similarity of three strongly random sets:

$$U \sim \mathcal{N} \sim \mathcal{MIN}.$$

Remark. It would be interesting to better understand the Borel subsets Σ of $(0, 1)_{\neq}^\infty / S_\infty$ having Λ measure one such that $\mathcal{MIN}(\alpha) \in \Sigma$ for all $\alpha \in KC$. In this paper we shall prove

Theorem 4 *There is a uniform procedure that, relative to a given $\alpha \in KC$, will yield, for any closed dyadic subinterval I of the unit interval, a sequence t_1, t_2, \dots of rationals in I that converges to the (unique) local minimizer of the complex oscillation, $\Phi(\alpha)$, in I . Moreover all the local minimizers of a complex oscillation are non-computable real numbers.*

We shall also prove

Theorem 5 *There is a Σ_3^0 predicate $C(\alpha, \nu)$ over $\{0, 1\}^\mathbb{N} \times \{0, 1\}^\mathbb{N}$ such that for $\alpha, \nu \in \{0, 1\}^\mathbb{N}$*

$$C(\alpha, \nu) \iff \nu \in \mathcal{MIN}(\Phi(\alpha)) \wedge \alpha \in KC.$$

This is a Σ_3^0 -representation, in effective descriptive set theory, of countably random dense sets, independent and generic as explained above and given by the minimizers of Brownian motions which are encoded by KC -strings.

Remark. By specialising to a Δ_2^0 -element Ω_0 in KC (a Chaitin real), we thus find a Σ_4^0 -predicate describing the local minimizers of the complex oscillation $\Phi(\Omega_0)$.

The proofs of these theorems appear in Section 4 of this paper.

3 Effective descriptions of Brownian motion

It is a daunting task to reflect sample path properties of Brownian motion (nowhere differentiability, law of the iterated algorithm, fractal geometry) into complex oscillations by defining these phenomena in terms of the basic events in B (see (4)) which is an effective basis for the uniform norm topology in $C[0, 1]$. For this reason the author introduced in [Fouché(1) 2000] another characterisation of the class \mathcal{C} by using basic descriptions relative to effective Boolean subalgebras of the Borel algebra on $C[0, 1]$.

In order to describe this characterisation, we follow [Fouché(1) 2000] to define an analogue of a Π_2^0 subset of $C[0, 1]$ which is of constructive measure 0. If F is a subset of $C[0, 1]$, we denote by \bar{F} its topological closure in $C[0, 1]$ with the uniform norm topology. For $\epsilon > 0$, we let $O_\epsilon(F)$ be the ϵ -ball $\{f \in C[0, 1] : \exists g \in F \|f - g\| < \epsilon\}$ of f . (Here $\|\cdot\|$ denotes the supremum norm.) We write F^0 for the complement of F and F^1 for F .

Definition 2 *A sequence $\mathcal{F}_0 = (F_i : i < \omega)$ in Σ is an effective generating sequence if*

1. *for $F \in \mathcal{F}_0$, for $\epsilon > 0$ and $\delta \in \{0, 1\}$, we have, for $G = O_\epsilon(F^\delta)$ or for $G = F^\delta$, that $W(\bar{G}) = W(G)$,*
2. *there is an effective procedure that yields, for each sequence $0 \leq i_1 < \dots < i_n < \omega$ and $k < \omega$ a binary rational number β_k such that*

$$|W(F_{i_1} \cap \dots \cap F_{i_n}) - \beta_k| < 2^{-k},$$

3. *for $n, i < \omega$, a strictly positive rational number ϵ and for $x \in C_n$, both the relations $x \in O_\epsilon(F_i)$ and $x \in O_\epsilon(F_i^0)$ are recursive in x, ϵ, i and n , relative to an effective representation of the rationals.*

Remark. This definition was motivated by the desire to have a class of basic statistical events, which, firstly, are relevant to the practice of Brownian motion, and, secondly, is such that one can prove an effective version of the Donsker invariance principle and therefore, thirdly, to capture the entire class of complex oscillations.

If $\mathcal{F}_0 = (F_i : i < \omega)$ is an effective generating sequence and \mathcal{F} is the Boolean algebra generated by \mathcal{F}_0 , then there is an enumeration $(T_i : i < \omega)$ of the elements of \mathcal{F} (with possible repetition) in such a way, for a given i , one can effectively describe T_i as a finite union of sets of the form

$$F = F_{i_1}^{\delta_1} \cap \dots \cap F_{i_n}^{\delta_n}$$

where $0 \leq i_1 < \dots < i_n$ and $\delta_i \in \{0, 1\}$ for each $i \leq n$. We call any such sequence $(T_i : i < \omega)$ a *recursive enumeration* of \mathcal{F} . We say in this case that \mathcal{F} is *effectively generated* by \mathcal{F}_0 and refer to \mathcal{F} as an *effectively generated algebra* of sets.

Let $(T_i : i < \omega)$ be a recursive enumeration of the algebra \mathcal{F} which is effectively generated by the sequence $\mathcal{F}_0 = (F_i : i < \omega)$ in Σ . It is shown in [Fouché(1) 2000] that there is an effective procedure that yields, for $i, k < \omega$, a binary rational β_k such that

$$|W(T_i) - \beta_k| < 2^{-k},$$

in other words, the function $i \mapsto W(T_i)$ is computable.

A sequence (A_n) of sets in \mathcal{F} is said to be \mathcal{F} -*semirecursive* if it is of the form $(T_{\phi(n)})$ for some total recursive function $\phi : \omega \rightarrow \omega$ and some effective enumeration (T_i) of \mathcal{F} . (Note that the sequence (A_n^c) , where A_n^c is the complement of A_n , is also an \mathcal{F} -semirecursive sequence.) In this case, we call the union $\bigcup_n A_n$ a $\Sigma_1^0(\mathcal{F})$ -set. A set is a $\Pi_1^0(\mathcal{F})$ -set if it is the complement of a $\Sigma_1^0(\mathcal{F})$ -set. It is of the form $\bigcap_n A_n$ for some \mathcal{F} -semirecursive sequence (A_n) . A sequence (B_n) in \mathcal{F} is a *uniform* sequence of $\Sigma_1^0(\mathcal{F})$ -sets if, for some total recursive function $\phi : \omega^2 \rightarrow \omega$ and some effective enumeration (T_i) of \mathcal{F} , each B_n is of the form

$$B_n = \bigcup_m T_{\phi(n,m)}.$$

In this case, we call the intersection $\bigcap_n B_n$ a $\Pi_2^0(\mathcal{F})$ -set. If, moreover, the Wiener-measure of B_n converges *effectively* to 0 as $n \rightarrow \infty$, we say that the set given by $\bigcap_n B_n$ is a $\Pi_2^0(\mathcal{F})$ -set of constructive measure 0.

The proof of the following theorem appears in [Fouché(1) 2000].

Theorem 6 *Let \mathcal{F} be an effectively generated algebra of sets. If x is a complex oscillation, then x is in the complement of every $\Pi_2^0(\mathcal{F})$ -set of constructive measure 0.*

This means, that every complex oscillation is, in an obvious sense, \mathcal{F} -Martin-Löf random. The converse is also true.

Definition 3 *An effectively generated algebra of sets \mathcal{F} is universal if the class \mathcal{C} of complex oscillations is definable by a single $\Sigma_2^0(\mathcal{F})$ -set, the complement of which is a set of constructive measure 0. In other words, \mathcal{F} is universal iff a continuous function x on the unit interval is a complex oscillation iff x is \mathcal{F} -Martin-Löf random.*

We introduce two classes of effectively generated algebras \mathcal{G} and \mathcal{M} which are very useful for reflecting properties of one-dimensional Brownian motion into complex oscillations.

Let \mathcal{G}_0 be a family of sets in Σ each having a description of the form:

$$a_1 X(t_1) + \dots + a_n X(t_n) \leq L \tag{6}$$

or of the form (6) with \leq replaced by $<$, where all the a_j, t_j ($0 \leq t_j \leq 1$) are non-zero computable real numbers, L is a recursive real number and X is one-dimensional Brownian motion.

We require that it be possible to find an enumeration $(G_i : i < \omega)$ of \mathcal{G}_0 such that, for given i , if G_i is given by (6), we can effectively compute the sign, and for every n , a rational approximation to each of a_j, t_j with error at most $1/n$.

As in [Fouché(2) 2000] it can be shown that $\mathcal{G}_0 = (G_i : i < \omega)$ is an effective generating sequence in the sense of Definition 2. The argument on p 325 of [Fouché(2) 2000] holds verbatim for this slight generalisation. The associated effectively generated algebra of sets \mathcal{G} will be referred to as a *gaussian algebra*.

It is shown in [Fouché(1) 2000] that if \mathcal{G}_0 is defined by events of the form (6) with $n = 1$ and $a_1 = 1$, then the associated \mathcal{G} is in fact *universal* in the sense of Definition 3.

For a closed subinterval I of the unit interval and a real number b , we write $[M(I) \geq b]$ for the event $[\sup\{X(t) : t \in I\} \geq b]$ and $[m(I) \leq b]$ for the event $[\inf\{X(t) : t \in I\} \leq b]$, where X is one-dimensional Brownian motion on the unit interval. We let \mathcal{M}_0 be the set of the events of the form $[M(I) \leq b]$ or $[m(I) \leq b]$ where b is an arbitrary rational number and where I is a subinterval of the unit interval with rational endpoints. It follows from the arguments on pp 434 - 438 in [Fouché(1) 2000] that the elements of \mathcal{M}_0 can be effectively enumerated rendering \mathcal{M}_0 an effective generating sequence. We denote by \mathcal{M} the Boolean algebra generated by \mathcal{M}_0 . It is shown in [Fouché(1) 2000] that \mathcal{M} too is in fact universal.

We shall also make frequent use of the following result from [Fouché(1) 2000] which is an easy consequence of Theorem 6. It is the analogue, for continuous functions, of the well-known fact that Kurtz-random reals contain the class of Martin-Löf random reals.

Theorem 7 *If B is a $\Sigma_1^0(\mathcal{F})$ set and $W(B) = 1$, then \mathcal{C} , the set of complex oscillations, is contained in B .*

4 Local minimizers of Brownian motion

In this section we shall prove Theorems 4 and 5. A crucial remark is that the local minimizers in subintervals of the unit interval of complex oscillations are uniquely determined:

Proposition 1 *If $x \in \mathcal{C}$, and I_1, I_2 are closed disjoint subintervals of the unit interval having rational endpoints, then*

$$\inf_{t \in I_1} x(t) \neq \inf_{t \in I_2} x(t).$$

The proof is a constructive version of the argument on p 20 of [Peres]. We shall need the following

Lemma 1 *Let Z_1, Z_2 be independent real-valued random variables on some probability space with Z_2 having a non-atomic distribution. Then, almost surely, $Z_1 + Z_2 \neq 0$.*

Proof. For $i = 1, 2$, write μ_i for the distribution measure of Z_i . Then $Z_1 + Z_2$ has the convolution product $\mu_1 * \mu_2$ as its distribution measure, which will be non-atomic when μ_2 is. Indeed, for a Borel set A of real numbers,

$$(\mu_1 * \mu_2)(A) = \int_{\mathbf{R}} \mu_2(A - t) d\mu_1(t),$$

and for $A = \{0\}$, it is the case that $\mu_2(A - t) = 0$ for all t (the measure μ_2 being non-atomic). The result follows since $\mu_1 * \mu_2(\{0\})$ is the probability of the event $[Z_1 + Z_2 = 0]$.

Proof of Proposition 1. It is well known (see p20 of [Peres]) that, under the hypotheses on I_1, I_2 , almost surely, $m(I_1) \neq m(I_2)$. Indeed, let $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ denote the lower and higher interval, respectively. Then the event $[m(I_1) = m(I_2)]$ is the same as

$$X(a_2) - X(b_1) = (m(I_1) - X(b_1)) - (m(I_2) - X(a_2)).$$

Since successive increments of Brownian motion are statistically independent, the random variable given by the expression on the right-hand side of this equation is independent from the random variable on the left-hand side while the latter is non-atomic, being absolutely continuous with respect to Lebesgue measure. It follows from the preceding lemma that $m(I_1) \neq m(I_2)$ almost surely.

The event $[m(I_1) \neq m(I_2)]$ is described by the following $\Sigma_1^0(\mathcal{M})$ event of Wiener measure one:

$$\exists_{r \in \mathbf{Q}} (m(I_1) < r < m(I_2)) \vee (m(I_2) < r < m(I_1)).$$

The proposition follows from Theorem 7 with $\mathcal{F} = \mathcal{M}$.

The proposition has the following

Corollary 1 *For every complex oscillation, for every dyadic interval I in the unit interval, there is a unique point in I where the minimum of x is assumed.*

We shall refer to this point as the *minimizer* of x in I . We shall also need

Lemma 2 *If $x \in \mathcal{C}$ and d_1, d_2 are distinct rational numbers in the unit interval, then $x(d_1) \neq x(d_2)$.*

Proof. Indeed, for a one-dimensional Brownian motion X , the random variable $X(d_1) - X(d_2)$ is normal with variance $|d_1 - d_2|$ and is therefore non-atomic being absolutely continuous with respect to Lebesgue measure. Consequently, almost surely, $X(d_1) \neq X(d_2)$.

Moreover, the almost sure event $[X(d_1) \neq X(d_2)]$ has a $\Sigma_1^0(\mathcal{G})$ description with respect to a suitable gaussian algebra \mathcal{G} . The description is given by the predicate

$$\exists_{r \in \mathbb{Q}^+} |x(d_1) - x(d_2)| > r$$

over $C[0, 1]$. Next apply Theorem 7 with $\mathcal{F} = \mathcal{G}$.

Remark. The preceding argument can very easily be adapted to show that each complex oscillation is *injective* when restricted to the computable reals in the unit interval.

A consequence of Proposition 1 is that for $x \in \mathcal{C}$, one can associate, with every dyadic subinterval I of the unit interval, the unique real number $\tau_I^x \in I$ which is the local minimizer of x in the interval I . By using this fact we shall now show that one can find a measurable mapping $\mathcal{N} : KC \rightarrow (0, 1)_{\neq}^\infty$ which upon composition with the projection $\pi : (0, 1)^\infty \rightarrow (0, 1)_{\neq}^\infty / S_\infty$ yields the strongly random set $\mathcal{MIN} : KC \rightarrow (0, 1)_{\neq}^\infty / S_\infty$.

To define \mathcal{N} we firstly note that it follows from Lévy's arcsine law that the distribution of the variables τ_I^x are all absolutely continuous with respect to Lebesgue measure. (See, for example [Peres].)

In fact, it follows from Lévy's arcsine law that, if $I = [a, b]$ and $a < \alpha < \beta < b$ then

$$W(\alpha < \tau_I^x < \beta) = \frac{1}{\pi} \int_{\frac{\alpha-a}{b-a}}^{\frac{\beta-a}{b-a}} \frac{dt}{\sqrt{t(1-t)}}.$$

In particular, for a recursive real number r in and a dyadic subinterval I of the unit interval, the event A given by

$$x \in A \Leftrightarrow \tau_I^x = r$$

is such that $W(A) = 0$. Moreover, writing \mathbb{D} for the set of dyadic rationals in the unit interval:

$$x \in A \Leftrightarrow \forall_{q \in \mathbb{D}} x(r) \leq x(q),$$

which means that A has a $\Pi_1^0(\mathcal{G})$ description relative to a suitable gaussian algebra \mathcal{G} . It again follows from Theorem 7 that A contains no complex oscillations.

Therefore, writing \mathbb{R}_r for the field of recursive real numbers, we have:

Theorem 8 *If $\alpha \in KC$ then*

$$\text{MIN}(\Phi(\alpha)) \cap \mathbb{R}_r = \emptyset.$$

We now define the mapping $\mathcal{N} : KC \rightarrow (0, 1)_{\neq}^\infty$ in stages. At stage 0 we select the unique minimizer of $\Phi(\alpha)$ in the unit interval. At stage n , we first select

$$(\tau_{[\frac{k}{2^n}, \frac{k+1}{2^n}]}^{\Phi(\alpha)} : 0 \leq k < 2^n)$$

and only add the local minimizers that haven't been selected at an earlier stage. The enumeration is well-defined for the minimizers are all, by Theorem 8, not endpoints of the intervals.

Remark. It would be interesting to know, whether, for $\alpha \in KC$ it is the case that

$$\text{MIN}(\Phi(\alpha)) \subset KC.$$

Proof of Theorem 4. In the sequel we shall, for $\alpha \in KC$, denote the function $\Phi(\alpha)$ also by x_α .

Let I be a fixed dyadic interval. For $n \geq 1$, set

$$D_n = \left\{ \frac{k}{2^n} : 0 \leq k < 2^n \wedge \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right] \subset I \right\}.$$

For $d \in D_n$, set $I_d = [\frac{k}{2^n}, \frac{k+1}{2^n})$, when $d = \frac{k}{2^n}$. Note that, for each $\beta \in I$, there is some N such that $\beta \in \cup_{d \in D_n} I_d$ for all $n \geq N$.

It follows from Lemma 2 and Theorem 3 that for $\alpha \in KC$ and dyadic rationals d_1, d_2 , the relation $x_\alpha(d_1) < x_\alpha(d_2)$ is decidable in α, d_1 , and d_2 . It is because we know that $x_\alpha(d_1) \neq x_\alpha(d_2)$. (Lemma 2.)

Fix $\alpha \in KC$ and write x_α for the associated complex oscillation. The sequence $T = (t_n)$ is computed by the prescription that for all n with D_n nonempty, we let t_n be the unique element of D_n such that $x_\alpha(t_n) < x_\alpha(d)$ for all $d \in D_n$ with $d \neq t_n$. In view of Theorem 3, the sequence T is computable from α .

Let (t_{n_k}) be any convergent subsequence of T with limit ν , say. For given $\eta > 0$, we have for all k sufficiently large ($\geq L$, say) that $\nu \in [t_{n_k} - \eta, t_{n_k} + \eta]$. Fix $\beta \in I$. Next choose $L_1 \geq L$ such for $k \geq L_1$ we can find some d_k in D_{n_k} such that $\beta \in I_{d_k}$.

For $k \geq L_1$

$$x_\alpha(\nu) - x_\alpha(\beta) = x_\alpha(\nu) - x_\alpha(t_{n_k}) + x_\alpha(t_{n_k}) - x_\alpha(d_k) + x_\alpha(d_k) - x_\alpha(\beta),$$

and, since, by construction,

$$x_\alpha(t_{n_k}) \leq x_\alpha(d_k),$$

the difference $x_\alpha(\nu) - x_\alpha(\beta)$ can be made to be arbitrarily small by first choosing η sufficiently small and then k sufficiently large. We conclude that $x_\alpha(\nu) \leq x_\alpha(\beta)$. In particular,

$$x_\alpha(\nu) = m(I).$$

Recall that x_α has a unique minimizer in I . Hence all the convergent subsequences of T have the same limit. We can therefore conclude that T is a convergent sequence converging to the unique point in I where the minimum of x_α on I is assumed. This concludes the proof of the theorem.

Remark. Even though the construction of the sequence $(t_i : i \geq 0)$ in the theorem is effective relative to $\alpha \in KC$, the proof renders no information on the rate of convergence to the local minimizer in the dyadic interval. This problem will be addressed in a sequel of this paper (in collaboration with George Davie where it will be shown how it can be uniformly computed from the incompressibility coefficient of $\alpha \in KC$).

We again write \mathbb{D} for the dyadic rationals in the unit interval. For the proof of the Theorem 5, we shall need the following

Proposition 2 *The relations $x_\alpha(\mu) < x_\alpha(t)$ and $x_\alpha(\mu) > x_\alpha(t)$ are each Σ_2^0 in $\alpha \in KC$, $\mu \in \{0, 1\}^{\mathbb{N}}$ and $t \in \mathbb{D}$.*

To prove this Proposition, we first discuss the following Lemma:

Lemma 3 *There is a uniform algorithm that, having access to an oracle for $\alpha \in KC$, will decide whether*

$$\Phi(\alpha)(t) < q,$$

for $t \in \mathbb{D}$ and $q \in \mathbb{R}_r$.

Proof: It follows from [Fouché(1) 2000] that, under the above hypotheses on α, t and q

$$\Phi(\alpha)(t) \neq q.$$

Since

$$\Phi(\alpha)(t) > q \Leftrightarrow \exists_n \overline{\Phi(\alpha)(t)}(n) > q,$$

the inequality $\Phi(\alpha)(t) > q$ can be algorithmically affirmed if true. (This a direct consequence of Theorem 3.)

To affirm the inequality $\Phi(\alpha)(t) < q$ we need only apply (5) and note that

$$\Phi(\alpha)(t) < q \Leftrightarrow \Phi(\hat{\alpha})(t) > -q,$$

to conclude the proof of the lemma. It is shown in [Fouché(2) 2000] that every complex oscillation is everywhere β -Hölder continuous for any $0 < \beta < \frac{1}{2}$. The proof of the Σ_2^0 -definability of the relation $x_\alpha(\mu) < x_\alpha(t)$ now follows from this observation together with Lemma 3 which allows one to infer that:

$$x_\alpha(\mu) < x_\alpha(t) \Leftrightarrow \exists_{q \in \mathbb{Q}} \exists_k \forall_{L \geq k} x_\alpha(\overline{\mu}(L)) < q + \frac{1}{2^{L/3}} \wedge q < x_\alpha(t).$$

The Σ_2^0 -definability of $x_\alpha(\mu) > x_\alpha(t)$ now follows from symmetry. (Replace α by $\hat{\alpha}$.)

Proof of Theorem 5. Using the notation in the proof of the preceding theorem, define, for μ in the Cantor space $\{0, 1\}^{\mathbb{N}}$, and for $\alpha \in KC$, the predicate $I_d(\mu, \alpha)$ to mean that the real in the unit interval corresponding to μ is the unique minimum on I_d of x_α . Here again, we have set $I_d = [\frac{k}{2^n}, \frac{k+1}{2^n})$, when $d = \frac{k}{2^n}$. Note that for $\alpha \in KC$.

$$I_d(\mu, \alpha) \Leftrightarrow \mu \in I_d \wedge \forall_{t \in I_d \cap \mathbb{D}} [x_\alpha(\mu) < x_\alpha(t)].$$

Since every local minimizer of a complex oscillation is a non-dyadic number, we can replace $[x_\alpha(\mu) < x_\alpha(t)]$ by $[x_\alpha(\mu) \leq x_\alpha(t)]$ in the definition of $I_d(\mu, \alpha)$. It therefore follows from Proposition 2 that the predicate $I_d(\mu, \alpha)$ is a Π_2^0 -formula in μ and α . Writing again $\mathcal{N}(\alpha)$ for the set of local minimizers of x_α , we find,

$$\mu \in \mathcal{N}(\alpha) \leftrightarrow \exists_{d \in \mathbb{D}} I_d(\mu, \alpha).$$

This is a Σ_3^0 -formula in μ and α . Finally note that the set KC is Σ_2^0 -definable.

Remark: In view of Tsirelson's Theorem [Tsirelson 2006] (see Theorem 2 above), it is an interesting problem, to characterise, for $x \in \mathcal{C}$, the Borel sets B of Lebesgue measure 0 that are disjoint from $\text{MIN}(x)$. This is not always the case. For instance, if $B = \{z\}$ where z is the minimum of x on the unit interval then of course B will intersect the local minimisers of x . On the other hand, if $B = Z_x$, the zero set of x , then B does have Lebesgue measure 0 and will be disjoint from $\text{MIN}(x)$. To see this, note that if X is a continuous version of one-dimensional Brownian motion, then, almost surely, no local minimum of X will be a zero of X . For otherwise, there will be a neighbourhood of some zero of X containing no other zeroes of X , which contradicts the well-known fact that the zero set of X is almost surely perfect (the zero set being, for instance, almost surely, a set of non-zero Hausdorff dimension). It follows that, we have, for each interval I in the unit interval, almost surely

$$\exists_{r \in \mathbb{Q}^+} m(I) < -r \vee m(I) > r.$$

This is, for each closed interval I with rational endpoints, a $\Sigma_1^0(\mathcal{M})$ event of full Lebesgue measure and is consequently reflected in every complex oscillation. We conclude that if x is a complex oscillation, then

$$Z_x \cap \text{MIN}(x) = \emptyset.$$

References

- [Asarin 1988] Asarin, E.A.: Individual random signals: an approach based on complexity, doctoral dissertation, Moscow State University, 1988
- [Asarin and Prokovskii 1986] Asarin, E. A. and Prokovskii, A. V.: Use of the Kolmogorov complexity in analysing control system dynamics, *Automat. Remote Control* **47** (1986) 2128. Translated from: Primeenie kolmogorovskoi slozhnosti k anлізу dinamiki upravlyemykh sistem, *Automatika i Telemekhanika (Automation Remote Control)* **1** (1986) 2533.
- [Camia, Fontes and Newman 2005] Camia, F., Fontes, L. R. G., Newman, C. M.: The scaling limit geometry of near-critical 2D percolation, available at arXiv:cond-mat/0510740v1, 2005.
- [Chaitin 1987] Chaitin, G. A.: *Algorithmic information theory*, Cambridge University Press, 1987.
- [Fontes, Isopi, Newman and Ravishankar 2003] Fontes, L. R. G., Isopi, M., Newman, C. M., Ravishankar K., (2003): The Brownian web: characterisation and convergence, available at arXiv:math.PR/0304119v1, 2003
- [Davie and Fouché 2012] Davie, G., Fouché, W. L.: Constructing a generic Brownian motion effectively from a random binary sequence, (submitted.)
- [Fouché(1) 2000] Fouché, W. L.: Arithmetical representations of Brownian motion I, *J. Symb. Logic* **65** (2000), 421-442.
- [Fouché(2) 2000] Fouché, W. L.: The descriptive complexity of Brownian motion, *Advances in Mathematics* **155**, (2000), 317-343
- [Fouché 2008] Fouché, W. L.: Dynamics of a generic Brownian motion : Recursive aspects, in: From Gödel to Einstein: Computability between Logic and Physics, *Theoretical Computer Science* **394**, (2008), 175-186.
- [Fouché 2009] Fouché, W. L.: Fractals generated by algorithmically random Brownian motion, K. Ambos-Spies, B. Löwe, and W. Merkle (Eds.): CiE 2009, LNCS 5635, pp. 208-217, 2009.
- [Gács 2005] Gács, P.: Uniform test of algorithmic randomness over a general space, *Theoretical Computer Science* **341** (2005) 91137.
- [Hinman 1978] Hinman, P. G.: *Recursion-theoretic hierarchies*, Springer-Verlag, New York, 1978.
- [Hoyrup and Rojas 2009] Hoyrup, M. , Rojas, C.: Computability of probability measures and Martin-Löf randomness over metric spaces, *Information and Computation* **207**, (2009), 830-847.
- [Kechris 1999] Kechris, A. S.: New directions in descriptive set theory, *Bull. Symb. Logic* **5** (1999), 161-174.
- [Kjos-Hanssen and Nerode 2009] Kjos-Hanssen, B. and Nerode, A.: Effective dimension of points visited by Brownian motion , *Theoretical Computer Science* **410**, (2009), 347-354.
- [Kjos-Hanssen and Szabados 2011] Kjos-Hanssen, B., and Szabados, T.: Kolmogorov complexity and strong approximation of Brownian motion, *Proc. Amer. Math. Soc.* **139**, (2011), 3307-3316.
- [Manin 2010] Manin, Y.I. : *A course in Mathematical Logic for Mathematicians*, Springer-Verlag, 2010.
- [Martin-Löf 1966] Martin-Löf, P.: The definition of random sequences, *Information and Control* **9** (1966), 602-619.
- [Nies 2008] Nies, A. : *Computability and randomness*, Oxford Logic Guides **51**, Clarendon Press, Oxford, 2008.

- [Peres] Peres, Y.: *An Invitation to Sample Paths of Brownian Motion*, available at www.stat.berkeley.edu/~peres/bmall.pdf
- [Potgieter 2012] Potgieter, Paul.: The rapid points of a complex oscillation, *Logical Methods in Computer Science* **1** (2012), 1-11.
- [Rudin 1960] Rudin, W.: *Fourier Analysis on Groups*, Interscience Publishers, New York - London, 1960.
- [Tsirelson 2004] Tsirelson, B.: Nonclassical stochastic flows and continuous products, *Probability Surveys* **1** (2004), 173-298. doi 10.1214/1549578041000000042.
- [Tsirelson 2006] Tsirelson, B.: Brownian local minima, random dense countable sets and random equivalence classes, *Electronic Journal of Probability* **11** (2006), 162-198.
- [Weihrauch 1999] Weihrauch, K.: Computability on the probability measures on the Borel sets of the unit interval, *Theoretical Computer Science* **219** (1999), 421-437.
- [Weihrauch 2000] Weihrauch, K.: *Computable Analysis*, Springer, Berlin, 2000.